IS THE KOLMOGOROFF MODEL APPLICABLE TO LARGE-SCALE TURBULENCE?

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Abstract. We discuss the difficulties encountered when the Heisenberg–Kolmogoroff model for turbulence is applied to the large-scale turbulence in: (A) molecular clouds (specifically the velocity vs size relationship) and (B) stars (specifically, the estimate of convective fluxes).

A new model for large-scale turbulence is, therefore, needed.

1. Introduction

The main difficulty encountered in constructing a model of turbulence lies in the well-known 'closure problem', whereby the equation for $\langle v^n \rangle$ (v is the fluctuating or turbulent velocity), depends on terms of the form $\langle v^{n+1} \rangle$ which in turn satisfy an equation involving $\langle v^{n+2} \rangle$, giving rise to an infinite chain of connected equations. For $\langle v^2 \rangle$, one has the well-known energy equation (cf. Batchelor, 1970)

$$\varepsilon(k) = \{\nu + \nu_t(k)\} \int_{k_0}^{k} 2k^2 F(k)\, \mathrm{d}k\,. \tag{1}$$

The energy $\varepsilon(k)$ (per unit mass and time) fed into the system in the interval $k_0 - k$ is partly *dissipated* by viscous forces $\sim \nu(\nabla v)^2 \sim \nu k^2 v^2$, and partly *transferred* to higher k by the nonlinear terms $\sim v^3$. Following the original suggestion by Heisenberg, the transfer process is written as the product of two terms. The first term represents the loss of energy by the eddies in the interval k_0 to k while the second term, represented by the action of a turbulent viscosity $\nu_t(k)$, describes the redeposition of the same energy to the eddies in the remaining interval from k to ∞,

$$\nu_t(k) = \int_{k}^{\infty} \nu_t^{(k)}\, \mathrm{d}k/k\,, \tag{2}$$

where $\nu_t^{(k)}$ represents the eddy viscosity exerted by turbulence on a band of wavenumbers centered around k. Clearly, the 'closure problem' is equivalent to prescribing the function $\nu_t^{(k)}$.

In (1), $F(k)$ is the energy spectral function, i.e., $\frac{1}{2}F(k)\, \mathrm{d}k$ is the energy contained in

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the wavenumber interval between k and $k + \mathrm{d}k$,

$$v^2(k) = \int_k^\infty F(k)\,\mathrm{d}k\,. \tag{3}$$

Integrating (1) over all k we obtain

$$\varepsilon = 2\nu \int_{k_0}^\infty k^2 F(k)\,\mathrm{d}k\,, \qquad \varepsilon \equiv \varepsilon(\infty) = \text{constant}\,, \tag{4}$$

which expresses the *global* energy conservation, i.e., the nonlinear interactions transfer energy without dissipation.

A theory of turbulence aims at predicting the function $F(k)$, i.e., how turbulent energy is distributed among eddies of different sizes. It is useful to visualize a turbulent medium as a conglomerate of eddies of sizes ranging from large eddies ($\approx$ dimension of the system itself), to eddies small enough (large k's) for molecular forces to operate. The dynamics of the large eddies is critically dependent on the nature of the energy feeding mechanism, i.e., on the structure of the function $\varepsilon(k)$, which may depend on magnetic fields B, rotation Ω, etc. In general, $\varepsilon = \varepsilon(k, B, \Omega, \ldots)$. The large eddies receive the stirring energy and transfer it, via the nonlinear interactions and without dissipation, to all the eddies of smaller sizes. Finally, there are very small eddies whose dynamics depends strongly on the nature of the kinematic viscosity.

Let us now consider Equation (1) for which we need two ingredients: $\nu_t(k)$ and $\varepsilon(k)$. The first successful model of turbulence was worked out independently by Heisenberg and Kolmogoroff in the late forties (Batchelor, 1970). HK selected a special group of eddies sufficiently removed from the energy source to be independent of the specific nature of the stirring mechanism, but at the same time not too close to the high k region where dissipative forces are most effective. The first assumption implies that in the HK region, the stirring energy has suffered many cascading processes, thereby losing memory of its specific nature. The only remaining feature is, therefore, the total energy – i.e.,

$$\varepsilon(k) = \varepsilon = \text{constant} \tag{5}$$

independent of k.

Let us now analyze $\nu_t(k)$. Since the HK eddies are at an intermediate distance from the two regions (low and high k) where forcing occurs, they can be considered 'freely' evolving. This means that their mean free path λ_k can be identified with their size $l_k \sim k^{-1}$. Since, in general,

$$\nu_t^{(k)} \sim \lambda_k v_k \tag{6}$$

and

$$v^2(k) = \int_k^\infty v_k^2\,\mathrm{d}k/k\,, \tag{7}$$

it follows that

$$\nu_t^{(k)} = \gamma k \sqrt{\frac{F}{k^3}}\,, \tag{8}$$

where γ is a numerical constant. Substituting Equations (5), (8), and (2) in (1), we obtain for $F(k)$ the well-known result (for $\nu \to 0$)

$$F(k) \sim \varepsilon^{2/3}\, k^{-5/3}\,, \tag{9}$$

known as the Kolmogoroff spectrum. Inserting (9) in (3), we are led to a velocity-size relation of the form

$$v(l) \sim l^{1/3}\,. \tag{10}$$

The Kolmogoroff spectrum has been repeatedly confirmed by experiments on medium to small size turbulence (Grant *et al.*, 1962).

2. Large-Scale Turbulence

Having described the physical assumptions underlying the only successful model of turbulence presently available, the question naturally arises: can the HK model be applied to describe the turbulence encountered in most astrophysical and geophysical phenomena, namely large-scale turbulence (LST)? Stated differently, how must the two HK assumptions

$$\varepsilon(k) = \text{constant}\,, \qquad \nu_t(k) = \gamma \int_k^\infty \sqrt{\frac{F(k)}{k^3}}\, \mathrm{d}k \tag{11}$$

be modified in order to describe LST?

The first assumption in (11) is the easiest one to tailor to the LST region. To do this, consider that Equation (1) represents the balance between four processes: (1) the energy gain from the source (e.g., in the case of thermally driven convection this would be the contribution from the buoyancy forces), (2) the energy losses due to molecular viscosity ν, (3) the energy losses due to heat conduction χ, and finally (4) the energy losses (or gains) due to the transfer of energy among different wavenumbers k.

Of the four, the first three can be accommodated by the linear theory, their net effect being represented by the growth rate $n_s(k)$

$$n_s(k) = n_s(k, \nu, \chi)\,. \tag{12}$$

The fourth process, which cannot be accommodated within the linear theory is written as a two-step process, as explained earlier. With this, Equation (1) can be generalized to the form

$$2 \int_{k_0}^{k} F(k) n_s(k)\, \mathrm{d}k = 2\nu_t(k) \int_{k_0}^{k} k^2 F(k)\, \mathrm{d}k\,, \tag{13}$$

where the viscosity term in (1) is now folded into the growth rate $n_s(k)$ and where the factor of 2 in the left-hand side of (13) arises because the energy is a quadratic function of the amplitude.

Next, consider the eddy viscosity $v_t(k)$. It is the purpose of this paper to show that the adoption of the Heisenberg–Kolmogoroff closure (11) for the LST region (as in the work of Ledoux *et al.*, 1961; and Yamaguchi, 1963) leads to unacceptable results and that a new closure is, therefore, needed.

3. Molecular Clouds

In the last few years, it has been demonstrated (Larson, 1981; Leung *et al.*, 1982; Fleck, 1983; Myers, 1983; Scalo, 1984; Dame *et al.*, 1984; Henriksen and Turner, 1984) that molecular clouds exhibit a turbulent behavior and that their velocity vs size relation is of the form

$$v(l) = v_0 \left(\frac{l}{l_0}\right)^{1/2}, \qquad v_0 \simeq 1 \text{ km s}^{-1}, \qquad l_0 \simeq 1 \text{ pc}. \tag{14}$$

It will be shown that the use of (14) in (13), together with the $v_t(k)$ given by (11) implies a growth rate $n_s(k)$ that does not correspond to any physical process known or suspected to operate in molecular clouds. Conversely, it will also be shown that the use of a physically acceptable $n_s(k)$ in (13) together with $v_t(k)$ given by (11) leads to a velocity vs size relation that does not reproduce the well-established relation (14).

To show the first point, substitute (14) into (3). The resulting $F(k)$ is then

$$F(k) \sim k^{-2}. \tag{15}$$

Taking the derivative of (13) and using (15), the resulting $n_s(k)$ has the form (with k in units of k_0)

$$n_s(k) \sim k^{-1/2}(1 - k/3). \tag{16}$$

A plot of this function is shown in Figure 1. Inspection of the behavior of the growth rates for the instabilities believed to operate in molecular clouds (see Elmegreen, 1982, for an extended analysis) shows that none of them resembles (16).

As an example of a possible instability operating in molecular clouds, consider the Rayleigh–Taylor (RT) instability for two fluids of densities ρ_1 and ρ_2. The general form of $n_s(k)$, including effects due to kinematic viscosity, has been worked out in detail by Chandrasekhar (1961), and the function $n_s(k)$ can be found in his Figures 106 and 107. The behavior is very different from that of Figure 1. In the case of an RT instability with a superimposed magnetic field **B** parallel to **g**, the growth rate $n_s(k)$ is given by Equation (209), Chapter X of Chandrasekhar (1961). In the limits $k \to 0$ and $k \to \infty$, the behaviour is given by

$$\begin{aligned} &k \to 0, \quad n_s(k) \sim k^{1/2}, \\ &k \to \infty, \quad n_s(k) \sim \text{constant}, \end{aligned} \tag{17}$$

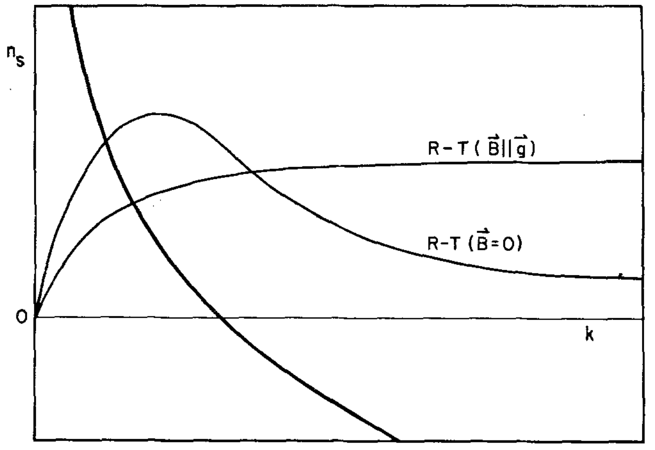

Fig. 1. Plot of growth rate n_s vs wavenumber k (arbitrary units). The curves labeled R–T are for the Rayleigh–Taylor instability with magnetic field **B** parallel to the gravitational acceleration **g** and for zero magnetic field. The boldface curve is the result of using velocity $v(l) \sim l^{1/2}$ in the HK theory.

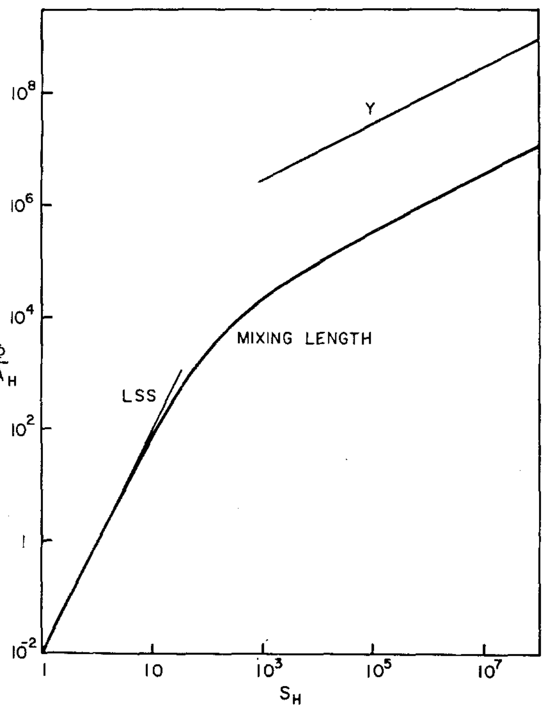

Fig. 2. Plot of convective flux Φ vs $S_H = \sigma R_H$. The boldface curve is the mixing length result calibrated using stellar models by Gough and Weiss (1976). The curves labeled LSS and Y are the results of Ledoux *et al.* (1961) and Yamaguchi (1963), respectively, based on the HK theory. The HK results are matched to the mixing length results in the low-flux limit. Note that the HK results are roughly two orders of magnitude too large in the large-flux limit.

again, quite different from that of Figure 2. Finally, in the case of an RT instability with a magnetic field **B** perpendicular to **g**, the form of $n_s(k)$ is given by (Chandrasekhar, 1961; Equation (234), Chapter X; A is a constant)

$$n_s(k) \sim k^{1/2}(1 - Ak)^{1/2}, \tag{18}$$

which again bears no resemblance to Figure 1.

Let us now analyze the second aspect of the HK closure, i.e., suppose a physically acceptable $n_s(k)$ is assumed. From the previous cases, it is clear that a constant growth rate can represent physically interesting cases (this is also true of the Parker (1967) instability when the physical parameters are chosen to correspond to the case of molecular clouds) (see Canuto and Battaglia, 1985). Substituting a constant n_s in Equation (13), one can solve for $F(k)$. The solution cannot be presented in closed analytical form, but the general behavior is of the form

$$F(k) \sim k^{-\alpha}, \tag{19}$$

with $\alpha \simeq 1$ up to $k \sim 12k_0$; azfter that $\alpha \simeq \frac{5}{3}$. In the first region, the result obtained from (3) is

$$v(l) \sim (\ln l)^{1/2}, \tag{20}$$

while in the second region

$$v(l) \sim l^{1/3}. \tag{21}$$

In the first interval, $v(l)$ given by (20) is almost constant, while in the second interval $v(l)$ grows with l only as $l^{0.33}$, while the data require a faster growth such as $l^{0.50}$ (cf. Turner, 1984).

4. Convective Fluxes

In this section, it will be shown that HK closure gives unacceptable results for convective turbulence fluxes. Since a theory of turbulent convection does not yet exist, the stellar model calculations, where an expression for the (turbulent) convective fluxes is required, have traditionally adopted a formula suggested by the Mixing Length Model (Gough, 1978; Cox and Giuli, 1968) which predicts (in units of the conductive flux)

$$\begin{aligned} S \ll 1:\quad \Phi &= AS^2, \qquad A = 8.75 \times 10^{-5}, \\ S \gg 1:\quad \Phi &= BS^{1/2}, \qquad B = 0.177, \end{aligned} \tag{22}$$

where

$$S = \beta\bar{\alpha}g\chi^{-2}l^4. \tag{23}$$

In Equation (23) l is the mixing length; β, the temperature gradient excess over the adiabatic gradient; g, the local gravity; χ, the thermometric conductivity and $\bar{\alpha}$, the thermal expansion coefficient. Using stellar models, Gough and Weiss (1976) calibrated the expressions (22) using as a calibrating parameter α,

$$l = \alpha H , \tag{24}$$

where H is the pressure scale-height. Equations (22) were, therefore, expressed as

$$\begin{aligned} S \ll 1: \quad \Phi &= A_H S_H^2 , \\ S \gg 1: \quad \Phi &= B_H S_H^{1/2} . \end{aligned} \tag{25}$$

For a solar metallicity of $Z = 0.02$, the value of α was found to be 1.1 thus yielding

$$A_H = 1.84 \times 10^{-4} , \qquad B_H = 0.214 . \tag{26}$$

Now consider the prediction of Equation (13). In the case of thermally-driven convection, the form of $n_s(k)$ is well-known ($k = k_0 q$)

$$n_s(k)/n_0 = \sqrt{1 + \lambda^2 q^4} - \lambda q^2 \tag{27}$$

(Ledoux *et al.*, 1961; Yamaguchi, 1963; Chandrasekhar, 1961) where for $\sigma \ll 1$, $2n_0 = \chi k_0^2 \lambda^{-1}$, and $\lambda^2 = 2\pi^4/S_d$; where $S_d = g\bar{\alpha}\beta d^4 \chi^{-2}$ is the product of σR, where σ is the Prandtl number and R is the Rayleigh number.

In the limits of interest here, Equation (13) was solved by Ledoux *et al.* (1960) for $S \ll 1$ and by Yamaguchi (1963) for $S \gg 1$. Their results for the convective fluxes are

$$\begin{aligned} S \ll 1: \quad \Phi &= \frac{1}{16\pi^{10}\gamma^2} S_d^2 , \\ S \gg 1: \quad \Phi &= \frac{3.2}{4\gamma^2} S_d^{1/2} , \end{aligned} \tag{28}$$

where d entering the theory through the growth rate, is the depth of the convective region. Writing d in terms of H,

$$d = \delta H , \tag{29}$$

the theoretical predictions can be expressed in a way directly comparable to (25) – i.e.,

$$\begin{aligned} S \ll 1: \quad \Phi &= \frac{\delta^8}{16\pi^{10}\gamma^2} S_H^2 , \\ S \gg 1: \quad \Phi &= \frac{3.2\delta^2}{4\gamma^2} S_H^{1/2} . \end{aligned} \tag{30}$$

The value of the parameter γ is about $\frac{1}{3}$ (Ledoux *et al.*, 1961). Suppose now that it is required that the theoretical formulae match expressions (25) in the *low*-flux region. In this case, δ is found to be $\delta = 1.53$, which in turn yields for the large-flux limit equation (30),

$$\Phi = 17 S_H^{1/2} , \tag{31}$$

a value 80 times larger than the calibrated value (25) and (26). The situation is illustrated in Figure 2.

Suppose now that the theoretical prediction is matched in the *high*-flux limit. One finds $\delta = 0.17$, yielding a low-flux limit of

$$\Phi = 4.7 \times 10^{-11} S_H , \tag{32}$$

which is seven orders of magnitude lower than the calibrated value (25)–(26). The situation is illustrated in Figure 3.

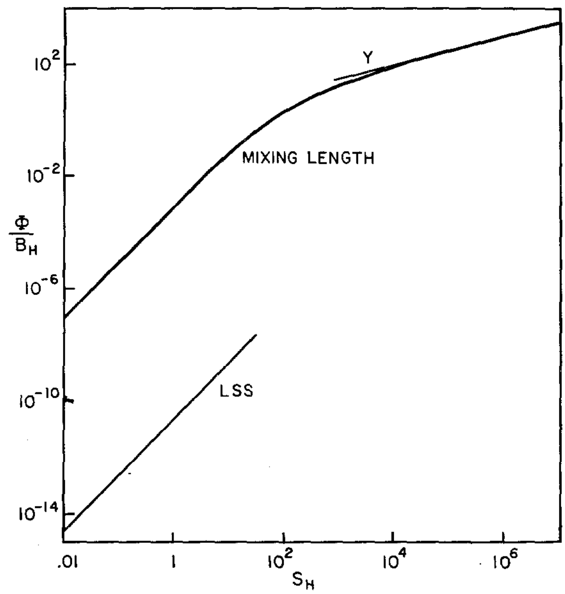

Fig. 3. Plot of convective flux Φ vs $S_H = \sigma R_H$. As in Figure 2, the HK results are matched to the mixing length results in the high-flux limit. Note that the HK results are roughly seven orders of magnitude too small in the small-flux limit.

Evidently, one has the option of *not* accepting the value of γ suggested by Ledoux *et al.* (1960) and consider it a free parameter. In this case, one can match *both* the low- and the high-flux limit provided one accepts that

$$\delta = 3.2 \quad \text{and} \quad \gamma = 6.3 . \tag{33}$$

In this case, a value of γ about twenty times larger than the standard value determined in laboratory turbulence would have to be justified.

It is, therefore, believed that the previous determination is the correct one, and that the HK closure does indeed lead to unacceptable results.

5. Conclusions

The Heisenberg–Kolmogoroff (HK) model of turbulence is often used to describe turbulent phenomena on all scales, though it is valid for a band of wavelengths in the turbulent spectrum that are typically much smaller than the size of the system.

Evidence has been presented here that the HK model cannot describe phenomena at large scales in astrophysical systems. First, the results of the mixing length theory cannot be accommodated within the framework of the HK model without adopting an unreasonable coupling constant.

Secondly, use of the observed velocity-size relationship in molecular clouds $v \sim l^{1/2}$ in the HK model, gives rise to a growth rate that does not correspond to any known physical process believed to operate in molecular clouds.

It is, therefore, concluded that a new model for large-scale turbulence is needed to describe these phenomena (for example, see Canuto and Goldman, 1985).

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